

Entropy evolution law in a laser process ^{*}

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Abstract

For the first time, we obtain the entropy variation law in a laser process after finding the Kraus operator of the master equation describing the laser process with the use of the entangled state representation. The behavior of entropy is determined by the competition of the gain and damping in the laser process. The photon number evolution formula is also obtained.

1 Introduction

Since the theoretical foundation proposed by Albert Einstein in 1917 ^[1] and the building of first functioning laser by Theodore H. Maiman in 1960, laser has been successfully applied in various of areas, including laser cooling technique developed by Steven Chu et al ^[2, 3]. As one of the most important concept in physics, entropy measures the disorder of a system. Studying the evolution of the entropy, we can get a clear understanding of how a laser beam is created by appropriate pumping. A few works had been done concerning the entropy exchange between a laser and its environment ^[4, 5]. However, the evolution of entropy in a laser itself has not yet been studied before. In this work we shall derive the entropy evolution law of a laser process. Our results explain how the

self-organization phenomenon happened in a laser.

In quantum optics theory the time evolution of laser in the lowest-order approximation can be described by the following master equation of density operator ^[6, 7, 8, 9]

$$\frac{d\rho(t)}{dt} = g [2a^\dagger \rho(t) a - aa^\dagger \rho(t) - \rho(t) aa^\dagger] + \kappa [2a \rho(t) a^\dagger - a^\dagger a \rho(t) - \rho(t) a^\dagger a], \quad (1)$$

where g and κ are the cavity gain and the loss, respectively, a^\dagger , a are photon creation and annihilation operator, respectively. It is also known that the evolution due to the interaction between a system and its environment can be ascribed to an evolution from the initial density operator ρ_0 to $\rho(t)$

$$\rho(t) = \sum_{n=0}^{\infty} M_n \rho_0 M_n^\dagger, \quad (2)$$

such an expression is named an operator-sum (Kraus) representation, M_n is named Kraus operator. So far as our knowledge is concerned, the entropy variation in laser-channel has not ever been reported. In this paper we shall show how the entropy of an initial coherent state $\rho_0 = |z\rangle\langle z|$ (the fact that n -photon distribution in a coherent state is Poisson distribution exactly fits the measurement result of photon distribution in a laser light) varies in the laser process, before doing this, we should first derive the Kraus operator by solving the master equation (1).

Our way is introducing the two-mode entangled state

$$|\eta\rangle = \exp(-\frac{1}{2}|\eta|^2 + \eta a^\dagger - \eta^* \tilde{a}^\dagger + a^\dagger \tilde{a}^\dagger) |0\tilde{0}\rangle, \quad (3)$$

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where \tilde{a}^\dagger is a fictitious mode independent of the real mode a^\dagger , $|\tilde{0}\rangle$ is annihilated by \tilde{a} , $[\tilde{a}, \tilde{a}^\dagger] = 1$. The state $|\eta = 0\rangle$ possesses the properties

$$\begin{aligned} a|\eta = 0\rangle &= \tilde{a}^\dagger|\eta = 0\rangle, \\ a^\dagger|\eta = 0\rangle &= \tilde{a}|\eta = 0\rangle, \\ (a^\dagger a)^n|\eta = 0\rangle &= (\tilde{a}^\dagger \tilde{a})^n|\eta = 0\rangle. \end{aligned} \quad (4)$$

Operating the both sides of (1) on the state $|\eta = 0\rangle \equiv |I\rangle$, and denoting $|\rho\rangle = \rho|I\rangle$, and using (4) we have the time-evolution equation for $|\rho(t)\rangle$,

$$\frac{d}{dt}|\rho(t)\rangle = \left[\begin{array}{c} g(2a^\dagger \tilde{a}^\dagger - aa^\dagger - \tilde{a}\tilde{a}^\dagger) \\ +\kappa(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger \tilde{a}) \end{array} \right] |\rho(t)\rangle. \quad (5)$$

where $|\rho_0\rangle \equiv \rho_0|I\rangle$, ρ_0 is the initial density operator.

The formal solution of (5) is

$$|\rho(t)\rangle = U(t)|\rho_0\rangle, \quad (6)$$

and

$$U(t) = \exp \left[\begin{array}{c} gt(2a^\dagger \tilde{a}^\dagger - aa^\dagger - \tilde{a}\tilde{a}^\dagger) \\ +\kappa t(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger \tilde{a}) \end{array} \right]. \quad (7)$$

It challenges us how to disentangle the exponential operator $U(t)$. This reminds us of two theorems about the normally ordered expansion of multimode bosonic exponential operators, which is helpful to disentangle $U(t)$.

2 Two Theorems

In order to find the disentangled form of (7) we employ two new theorems about the normally ordered expansion of multimode bosonic exponential operators [10, 11]:

Theorem 1: The multimode bosonic exponential operator $\exp \mathcal{H}$, where $\mathcal{H} = \frac{1}{2}B\Gamma\tilde{B}$, B is defined by

$$\begin{aligned} B &\equiv (A^\dagger A) \equiv \begin{pmatrix} a_1^\dagger & a_2^\dagger & \cdots & a_n^\dagger & a_1 & a_2 & \cdots & a_n \end{pmatrix} \\ \tilde{B} &= \begin{pmatrix} \tilde{A}^\dagger \\ \tilde{A} \end{pmatrix}, \end{aligned} \quad (8)$$

Γ is a $2n \times 2n$ matrix, has its n -mode coherent state representation:

$$\exp \mathcal{H} = \sqrt{\det Q} \int \prod_{i=1}^n \frac{d^2 Z_i}{\pi} \left| \begin{pmatrix} Q & -L \\ -N & P \end{pmatrix} \begin{pmatrix} \tilde{Z} \\ \tilde{Z}^* \end{pmatrix} \right\rangle \langle \begin{pmatrix} \tilde{Z} \\ \tilde{Z}^* \end{pmatrix} |, \quad (9)$$

here

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_2^2 = I_2, \quad (17)$$

where the n -mode coherent state is defined as

$$\begin{aligned} \left| \begin{pmatrix} \tilde{Z} \\ \tilde{Z}^* \end{pmatrix} \right\rangle &\equiv |Z\rangle = D(Z)|\tilde{0}\rangle, \\ D(Z) &\equiv \exp\{A^\dagger \tilde{Z} - A\tilde{Z}^*\}, \end{aligned} \quad (10)$$

and

$$\begin{pmatrix} Q & L \\ N & P \end{pmatrix} = \exp\{\Gamma\Pi\}, \quad \Pi = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \quad (11)$$

I_n is the $n \times n$ unit matrix. Q, L, N, P are $n \times n$ complex matrices, $\begin{pmatrix} Q & L \\ N & P \end{pmatrix} \equiv M$ is a symplectic matrix, obeying

$$M\Pi\tilde{M} = \Pi, \quad \Pi\tilde{M}\Pi = -M^{-1}, \quad (12)$$

or

$$\begin{aligned} Q\tilde{L} &= L\tilde{Q}, \quad Q\tilde{P} - L\tilde{N} = I, \\ N\tilde{P} &= P\tilde{N}, \quad P\tilde{Q} - N\tilde{L} = I. \end{aligned} \quad (13)$$

Theorem 2: By performing the integration in (9) with the technique of integration within an ordered product of operators [11, 12], we have

$$\begin{aligned} \exp \mathcal{H} &= \frac{1}{\sqrt{\det P}} \exp\{-\frac{1}{2}A^\dagger(LP^{-1})\tilde{A}^\dagger\} \\ &\times \exp\{A^\dagger(\ln P^{-1})\tilde{A}\} \exp\{\frac{1}{2}A(P^{-1}N)\tilde{A}\}. \end{aligned} \quad (14)$$

Now we first appeal to Theorem 1, so we should identify $U(t)$ in (7) as $\exp \mathcal{H}$, where $A = \begin{pmatrix} \tilde{a} & a \end{pmatrix}$. After putting $U(t)$ into the following symmetrized matrix form

$$U(t) = e^{(\kappa-g)t} \exp \left[\frac{1}{2}B\Gamma\tilde{B} \right] \quad (15)$$

with Γ being the symmetric matrix

$$\Gamma = t \begin{pmatrix} 2gJ_2 & -(g+\kappa)I_2 \\ -(g+\kappa)I_2 & 2\kappa J_2 \end{pmatrix} \quad (16)$$

we then follow (11) to calculate $\exp(\Gamma\Pi)$ with

$$\Gamma\Pi = t \begin{pmatrix} -(g+\kappa)I_2 & -2gJ_2 \\ 2\kappa J_2 & (g+\kappa)I_2 \end{pmatrix} \quad (18)$$

therefore

$$e^{\Gamma\Pi} \equiv \begin{pmatrix} Q & L \\ N & P \end{pmatrix} \quad (19)$$

with

$$\begin{aligned} Q &\equiv \frac{ge^{(\kappa-g)t} - \kappa e^{(g-\kappa)t}}{g-\kappa} I_2, \quad L \equiv \frac{g[e^{(\kappa-g)t} - e^{(g-\kappa)t}]}{g-\kappa} J_2, \\ N &\equiv \frac{\kappa[e^{(g-\kappa)t} - e^{(\kappa-g)t}]}{g-\kappa} J_2, \quad P \equiv \frac{ge^{(g-\kappa)t} - \kappa e^{(\kappa-g)t}}{g-\kappa} I_2. \end{aligned} \quad (20)$$

Thus according to Theorems 1 and 2 we have

$$\begin{aligned} U(t) &= \frac{\kappa-g}{\kappa e^{-2(g-\kappa)t} - g} \exp \left[\frac{g[1 - e^{-2(\kappa-g)t}]}{\kappa - ge^{-2(\kappa-g)t}} \tilde{a}^\dagger a^\dagger \right] \\ &\times \exp \left[(\tilde{a}^\dagger \tilde{a} + a^\dagger a) \ln \frac{(\kappa-g)e^{-(\kappa-g)t}}{\kappa - ge^{-2(\kappa-g)t}} \right] \\ &\times \exp \left[\frac{\kappa[1 - e^{-2(\kappa-g)t}]}{\kappa - ge^{-2(\kappa-g)t}} \tilde{a} \tilde{a} \right], \end{aligned} \quad (21)$$

where we have used

$$\begin{aligned} LP^{-1} &= \frac{g[1 - e^{-2(\kappa-g)t}]}{ge^{-2(\kappa-g)t} - \kappa} J_2, \\ P^{-1}N &= \frac{\kappa[1 - e^{-2(\kappa-g)t}]}{ge^{-2(\kappa-g)t} - \kappa} J_2 \end{aligned} \quad (22)$$

and

$$\sqrt{\det P} \equiv \frac{ge^{(g-\kappa)t} - \kappa e^{(\kappa-g)t}}{g-\kappa}. \quad (23)$$

writing

$$\begin{aligned} T_1 &= \frac{1 - e^{-2(\kappa-g)t}}{\kappa - ge^{-2t(\kappa-g)}}, \quad T_2 = \frac{(\kappa-g)e^{-(\kappa-g)t}}{\kappa - ge^{-2t(\kappa-g)}} \\ T_3 &= \frac{\kappa-g}{\kappa - ge^{-2t(\kappa-g)}} = 1 - gT_1, \end{aligned} \quad (24)$$

and using (4), (6) becomes

$$\begin{aligned} &|\rho(t)\rangle = U(t)|\rho_0\rangle \\ &= T_3 e^{gT_1 a^\dagger \tilde{a}^\dagger} : e^{(T_2-1)(\tilde{a}^\dagger \tilde{a} + a^\dagger a)} : e^{\kappa T_1 a \tilde{a}} |\rho_0\rangle \\ &= \sum_{i,j=0}^{\infty} T_3 \frac{\kappa^i g^j T_1^{i+j}}{i!j!T_2^{2j}} e^{a^\dagger a \ln T_2} a^{\dagger j} a^i \rho_0 a^{\dagger i} a^j e^{a^\dagger a \ln T_2} |\eta=0\rangle, \end{aligned} \quad (25)$$

or

$$\rho(t) = \sum_{i,j=0}^{\infty} M_{ij} \rho_0 M_{ij}^\dagger, \quad (26)$$

where

$$M_{ij} = \sqrt{\frac{\kappa^i g^j T_3 T_1^{i+j}}{i!j!T_2^{2j}}} e^{a^\dagger a \ln T_2} a^{\dagger j} a^i \quad (27)$$

is the Kraus operator, and one can check

$$\sum_{i,j=0}^{\infty} M_{ij}^\dagger M_{ij} = 1. \quad (28)$$

3 The Photon Number Evolution

Now we have an explicit solution of the density matrix of a laser (26), we first calculate the evolution of photon number in a laser process initially in a pure coherent state, i.e., $\rho_0 = |z\rangle\langle z|$, $|z\rangle = \exp[-|z|^2/2 + za^\dagger]|0\rangle$,

$$\begin{aligned} \langle n \rangle &= \text{Tr} [\rho(t) a^\dagger a] \\ &= e^{\kappa T_1 |z|^2} \text{Tr} \left[\sum_{j=0}^{\infty} T_3 \frac{g^j T_1^j}{j!T_2^{2j}} e^{a^\dagger a \ln T_2} a^{\dagger j} |z\rangle\langle z| a^j e^{a^\dagger a \ln T_2} a^\dagger a \right] \end{aligned} \quad (29)$$

Then using $|0\rangle\langle 0| =: e^{-a^\dagger a}:$, the normal ordering form of the vacuum projector, as well as $\int \frac{d^2 z}{\pi} |z\rangle\langle z| = 1$ we have the expected photon number evolution formula

$$\begin{aligned} \langle n \rangle &= T_3 e^{(\kappa T_1 - 1)|z|^2} \text{Tr} \left[\sum_{j=0}^{\infty} \frac{g^j T_1^j}{j!} a^{\dagger j} e^{z T_2 a^\dagger} : e^{-a^\dagger a} : e^{z^* T_2 a} a^j a^\dagger a \right] \\ &= T_3 e^{(\kappa T_1 - 1)|z|^2} \text{Tr} \left[e^{z T_2 a^\dagger} e^{a^\dagger a \ln(g T_1)} e^{z^* T_2 a} a^\dagger a \right] \\ &= T_3 e^{(\kappa T_1 - 1)|z|^2} \text{Tr} \left[e^{z T_2 a^\dagger} (g T_1 a^\dagger + z^* T_2) e^{a^\dagger a \ln(g T_1)} e^{z^* T_2 a} a \right] \\ &= T_3 e^{(\kappa T_1 - 1)|z|^2} \int \frac{d^2 z'}{\pi} \langle z' | \\ &\times : e^{z T_2 a^\dagger + z^* T_2 a + (g T_1 - 1)a^\dagger a} (g T_1 a^\dagger + z^* T_2) a : |z'\rangle \\ &= g \frac{1 - e^{-2(\kappa-g)t}}{\kappa-g} + |z|^2 e^{-2(\kappa-g)t}. \end{aligned} \quad (30)$$

We can easily write down the asymptotic behavior of $\langle n \rangle$ when $t \rightarrow +\infty$ as the following:

If $\kappa = g$, then $\langle n \rangle = |z|^2 + 2gt$, the photon number increases linearly with time.

If $\kappa < g$, then $\langle n \rangle \sim \left(\frac{g}{g-\kappa} + |z|^2 \right) e^{2(g-\kappa)t}$, the photon number increases exponentially when $t \rightarrow +\infty$.

If $\kappa > g$, then $\langle n \rangle \sim \frac{g}{\kappa-g}$, the expected photon number approaches a constant when $t \rightarrow +\infty$.

4 The Entropy Evolution in a Laser

We now calculate how the entropy of a laser evolves with time. Using (26), the density matrix $\rho(t)$ of a laser initially in a pure coherent state $|z\rangle\langle z|$ is

$$\begin{aligned} \rho(t) &= T_3 \exp \left[|z|^2 e^{2(g-\kappa)t} \ln gT_1 \right] \\ &\times \sum_{j=0}^{\infty} \frac{g^j T_2^j}{j! T_2^{2j}} : a^\dagger j a^j e^{z a^\dagger + z^* a - a^\dagger a} : e^{a^\dagger a \ln T_2} \\ &= T_3 e^{\kappa T_1 |z|^2 - |z|^2} e^{z T_2 a^\dagger} e^{a^\dagger a \ln(gT_1)} e^{z^* T_2 a}. \end{aligned} \quad (31)$$

By the Baker–Campbell–Hausdorff formula, if

$$[X, Y] = \lambda Y + \mu, \quad (32)$$

then

$$\exp X \exp Y = \exp \left(X + \frac{\lambda Y + \mu}{1 - e^{-\lambda}} - \frac{\mu}{\lambda} \right). \quad (33)$$

we can compact the three exponentials in (31) into a single exponential

$$\begin{aligned} \rho(t) &= T_3 \exp \left[|z|^2 e^{2(g-\kappa)t} \ln gT_1 \right] \\ &\times \exp \left\{ [a^\dagger a - e^{(g-\kappa)t} (za^\dagger + z^* a)] \ln gT_1 \right\}. \end{aligned} \quad (34)$$

with direct calculations. Thus we see how a pure state $|z\rangle\langle z|$ evolves into a mixed state, so the entangled state representation in (3) can well expose the entanglement between the system and its environment. Then the logarithm of $\rho(t)$ can be evaluated as

$$\begin{aligned} \ln \rho(t) &= \ln T_3 + |z|^2 e^{2(g-\kappa)t} \ln gT_1 \\ &+ [a^\dagger a - e^{(g-\kappa)t} (za^\dagger + z^* a)] \ln gT_1. \end{aligned} \quad (35)$$

Therefore the von-Neumann entropy of $\rho(t)$ is

$$\begin{aligned} S(\rho(t)) / k_B &= -\text{Tr} [\rho \ln \rho] \\ &= -\text{Tr} \left[\rho \left(\ln T_3 + |z|^2 e^{2(g-\kappa)t} \ln gT_1 \right) \right] - T_3 e^{(\kappa T_1 - 1)|z|^2} \ln gT_1 \\ &\quad \times \text{Tr} \left[e^{z T_2 a^\dagger} e^{a^\dagger a \ln gT_1} e^{z^* T_2 a} (a^\dagger a - e^{(g-\kappa)t} (za^\dagger + z^* a)) \right] \\ &= -\ln T_3 - |z|^2 e^{2(g-\kappa)t} \ln gT_1 - T_3 e^{(\kappa T_1 - 1)|z|^2} \ln gT_1 \\ &\quad \times \text{Tr} \left[e^{z T_2 a^\dagger} e^{a^\dagger a \ln gT_1} e^{z^* T_2 a} (a^\dagger a - e^{(g-\kappa)t} (za^\dagger + z^* a)) \right], \end{aligned} \quad (36)$$

where k_B is the Boltzmann constant. Since

$$\begin{aligned} &e^{z T_2 a^\dagger} e^{a^\dagger a \ln gT_1} e^{z^* T_2 a} [a^\dagger a - e^{(g-\kappa)t} (za^\dagger + z^* a)] \\ &= e^{z T_2 a^\dagger} e^{a^\dagger a \ln gT_1} (a^\dagger + z^* T_2) e^{z^* T_2 a} a \\ &\quad - e^{(g-\kappa)t} z e^{z T_2 a^\dagger} e^{a^\dagger a \ln gT_1} (a^\dagger + z^* T_2) e^{z^* T_2 a} \\ &= e^{z T_2 a^\dagger} (gT_1 a^\dagger + z^* T_2) e^{a^\dagger a \ln gT_1} e^{z^* T_2 a} a \\ &\quad - e^{(g-\kappa)t} z e^{z T_2 a^\dagger} (gT_1 a^\dagger + z^* T_2) e^{a^\dagger a \ln gT_1} e^{z^* T_2 a}, \end{aligned} \quad (37)$$

therefore

$$\begin{aligned} &\text{Tr} \left[e^{z T_2 a^\dagger} e^{a^\dagger a \ln gT_1} e^{z^* T_2 a} (a^\dagger a - e^{(g-\kappa)t} (za^\dagger + z^* a)) \right] \\ &= \int \frac{d^2 z'}{\pi} \langle z' | : e^{z T_2 a^\dagger + z^* T_2 a + (gT_1 - 1)a^\dagger a} \\ &\quad \times [(gT_1 a^\dagger + z^* T_2) (a - e^{(g-\kappa)t} z) - e^{(g-\kappa)t} z^* a] : |z'\rangle \\ &= \int \frac{d^2 z'}{\pi} e^{z T_2 z'^* + z^* T_2 z' + (gT_1 - 1)|z'|^2} \\ &\quad \times [(gT_1 z'^* + z^* T_2) (z' - e^{(g-\kappa)t} z) - e^{(g-\kappa)t} z^* z']. \end{aligned} \quad (38)$$

Finally the entropy variation law is

$$S(\rho(t)) = -k_B \left(\ln T_3 + \frac{gT_1 \ln gT_1}{1 - gT_1} \right) \quad (39)$$

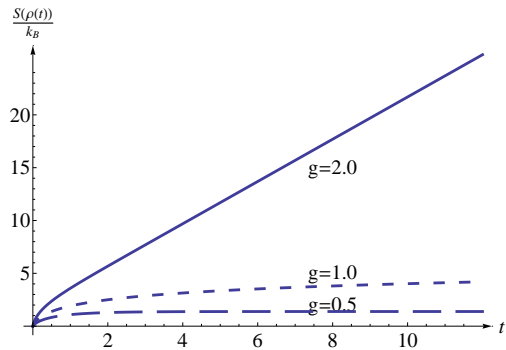


Figure 1: $S(\rho(t)) / k_B$ for $z = 4, \kappa = 1$ and $g = 2, 1, 0.5$ respectively

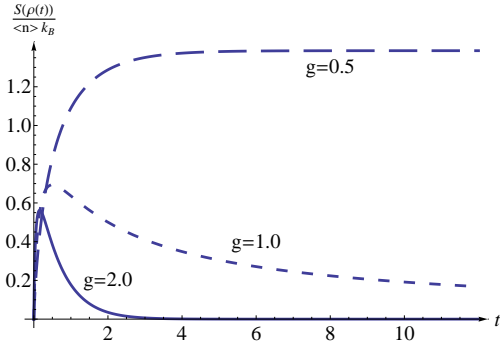


Figure 2: $S(\rho(t)) / (\langle n \rangle k_B)$ for $z = 4, \kappa = 1$ and $g = 2, 1, 0.5$ respectively

We also write down the asymptotic behavior of $S(\rho(t))$ when $t \rightarrow +\infty$ as the following:

If $\kappa = g$, then $S(\rho(t)) / k_B \sim 1 + \ln(2gt)$ as $t \rightarrow +\infty$, the entropy increases logarithmically.

If $\kappa < g$ then $S(\rho(t)) / k_B \sim 1 + \ln \frac{g}{g-\kappa} + 2(g-\kappa)t$ as $t \rightarrow +\infty$, the entropy increases linearly.

If $\kappa > g$ then $S(\rho(t)) / k_B \sim \ln \frac{\kappa}{\kappa-g} + \frac{g}{\kappa-g} \ln \frac{\kappa}{g}$ as $t \rightarrow +\infty$, the entropy approaches a constant.

The results of expected photon number and entropy of the laser do not depend on the phase of parameter z , as one should expect, since the absolute phase of z in a coherent state is non-physical. It is remarkable that the entropy is completely independent of z .

Plots of $S(\rho(t))$ and the "specific entropy" $\frac{S(\rho(t))}{\langle n \rangle}$ in unit of k_B for $z = 4$ (16 photons in average), $\kappa = 1$ and $g = 2, 1, 0.5$ respectively are shown in Figure 1 and 2. Besides the We can see clearly from the two figures that when the pumping rate g is less than the loss rate κ , the photon number and entropy will approach to constants, the photons are in fact in sort of thermo-equilibrium with an equivalent temperature $T = \frac{\hbar\omega}{k_B} \ln \frac{\kappa}{g}$. When g is larger than κ , while the entropy increases linearly with time, the expected number of photons increases much more fast, therefore the specific entropy will goes to zero exponentially. The photons in the laser are highly coherent in this case. The above results indicate that a laser can generate laser beam only if it works with sufficiently high pumping rate g .

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